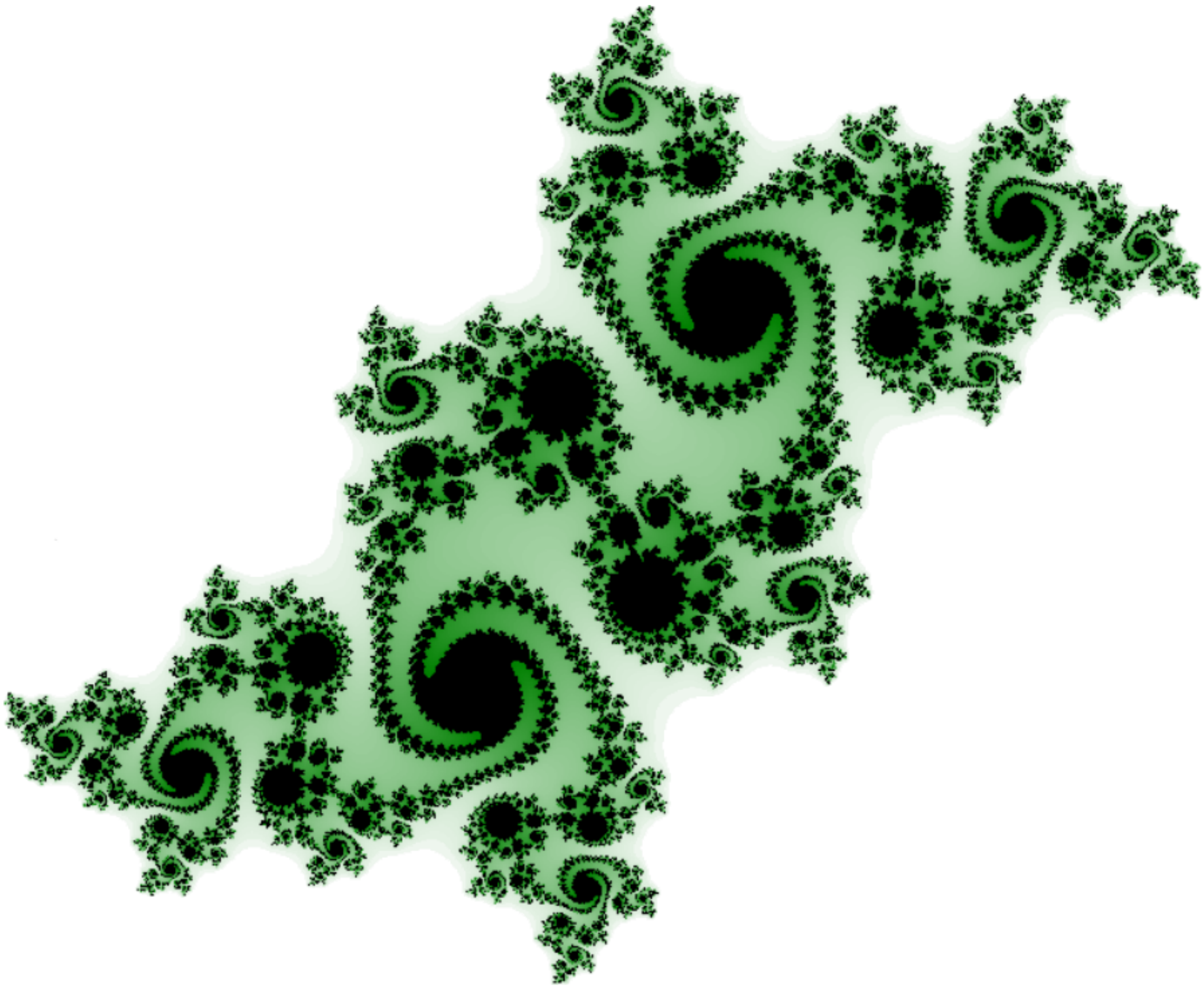

Gentle Introduction to Fractals

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Contents

1 Fractals Basics

Prior to explaining the usage of the interactive graphical application Fractal Explorer in NCLab, let us briefly summarize the concept, history, and mathematical foundation of fractals.

1.1 Concept

The word "fractal" often has different connotations for laypeople than mathematicians, where the layperson is more likely to be familiar with *fractal art* than a mathematical concept. The mathematical concept is difficult to formally define even for mathematicians, but key features can be understood with little mathematical background.

The feature of "self-similarity", for instance, is easily understood by analogy to zooming in with a lens or other device that zooms in on digital images to uncover finer, previously invisible, new structure. If this is done on fractals, however, no new detail appears; nothing changes and the same pattern repeats over and over, or for some fractals, nearly the same pattern reappears over and over. Self-similarity itself is not necessarily counter-intuitive (e.g., people have pondered self-similarity informally such as in the infinite regress in parallel mirrors or the homunculus, the little man inside the head of the little man inside the head...). The difference for fractals is that the pattern reproduced must be *detailed*.

Understanding this concept is essential for understanding fractals, so let us explain it in more detail. The following sequence of geometries starts with an equilateral triangle on the left. The next geometry to the right is obtained via splitting each boundary edge of the previous geometry into three equally-long pieces of length L. Then all the middle pieces are removed, and always two new edges of length L are added back, to add a new equilateral triangle to the missing middle section of each edge.

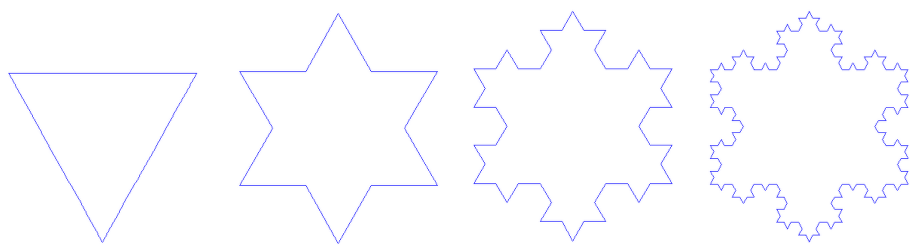

Figure 1: First four Von Koch's curves.

Exactly the same procedure is applied to get from the second geometry to the third, and from the third to the fourth. These geometries are called *von Koch's curves* but they are not fractals yet. In order to get a fractal, one has to do this infinitely many times, and take the limit.

Can you show that the boundary of the limit geometry has an infinite length?

This is quite interesting, given that obviously the area of the resulting fractal is finite. It also explains that fractals as mathematical objects are "nowhere differentiable". In a concrete sense, this means fractals cannot be measured in traditional ways. To elaborate, in trying to find the length of a wavy non-fractal curve, one could find straight segments of some measuring tool small enough to lay end to end over the waves, where the pieces could get small enough to be considered to conform to the curve in the normal manner of measuring with a tape measure. But in measuring a wavy fractal curve, one would never find a small enough straight segment to conform to the curve, because the wavy pattern would always re-appear, albeit at a smaller size, essentially pulling a little more of the tape measure into the total length measured each time one attempted to fit it tighter and tighter to the curve.

1.2 History

The mathematical roots of the idea of fractals have been traced through a formal path of published works, starting in the 17th century with notions of recursion, then moving through increasingly rigorous mathematical treatment of the concept to the study of continuous but not differentiable functions in the 19th century, and on to the coining of the word fractal in the 20th century with a subsequent burgeoning of interest in fractals and computer-based modelling in the 21st century [5, 6]. The term "fractal" was first used by mathematician Benoit Mandelbrot in 1975. Mandelbrot based it on the Latin *fractus* meaning "broken" or "fractured", and used it to extend the concept of theoretical fractional dimensions to geometric patterns in nature.

1.3 Mathematical Foundation

Mathematically, a fractal is a mathematical set that has a fractal dimension that usually exceeds its topological dimension [2] and may fall between the integers [3]. Fractals are typically self-similar patterns, where self-similar means they are "the same from near as from far" [4]. Fractals may be exactly the same at every scale, or they may be nearly the same at different scales. The definition of fractal goes beyond self-similarity per se to exclude trivial self-similarity and include the idea of a detailed pattern repeating itself.

Widely studied fractal sets are the Mandelbrot, Julia, and Newton sets.

Mandelbrot set
The Mandelbrot set consists of all values c in the complex plane for which the limit of the sequence

$$z_{n+1} = z_n^2 + c$$

with the starting value $z_0 = 0$ remains bounded.

Julia set
Let's take an arbitrary rational complex function $f(z)$ of the form

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are complex polynomials. The Julia set, usually called $J(f)$, is a set of all complex points z_0 for which the infinite sequence

$$z_{n+1} = f(z_n)$$

is bounded. Currently, NCLab uses the function

$$f(z) = z^2 + c$$

where $c = c_r + ic_i$ is a complex number defined using two real constants c_r and c_i. More general definition of f will be enabled in the future.

Newton set
The Newton set is a special case of the Julia set, defined for an arbitrary non-constant complex polynomial $h(z)$ via

$$z_{n+1} = z_n - \frac{h(z)}{h'(z)}$$

(note that this formula corresponds to the Newton's method used to solve nonlinear equations of the form $h(z) = 0$). In other words, the polynomials $p(z)$ and $q(z)$ that appear in the definition of the Julia set are defined as

$$p(z) = z_n h'(z) - h(z), \quad q(z) = h'(z).$$

Next let us mention some basic references on fractals. The presentation will be continued after that with the description of the Fractal Explorer.

References Cited

[1] Wikipedia: http://wikipedia.org.

[2] Mandelbrot, Benoit (2004). Fractals and Chaos. Berlin: Springer. ISBN 9780387201580.

[3] Mandelbrot,Benoit B. (1983). The fractal geometry of nature. Macmillan. ISBN 978-0-7167-1186-5. Retrieved 1 February 2012.

[4] Gouyet, J-F (1996). Physics and fractal structures. Paris New York: Masson Springer. ISBN 9780387941530.

[5] Edgar, Gerald (2004). Classics on Fractals. Boulder: Westview Press. ISBN 9780813341538.

[6] Trochet, Holly (2009). "A History of Fractal Geometry". MacTutor History of Mathematics. Archived from the original on 4 February 2012. Retrieved 4 February 2012.

2 Fractal Explorer

The Fractal Explorer in NCLab allows the users to get familiar with the structure of the Mandelbrot and Julia sets (Newton set is in preparation). It can be launched by clicking on the *Fractals* icon on the desktop. The following Fig. 2 shows the classical Mandelbrot set.

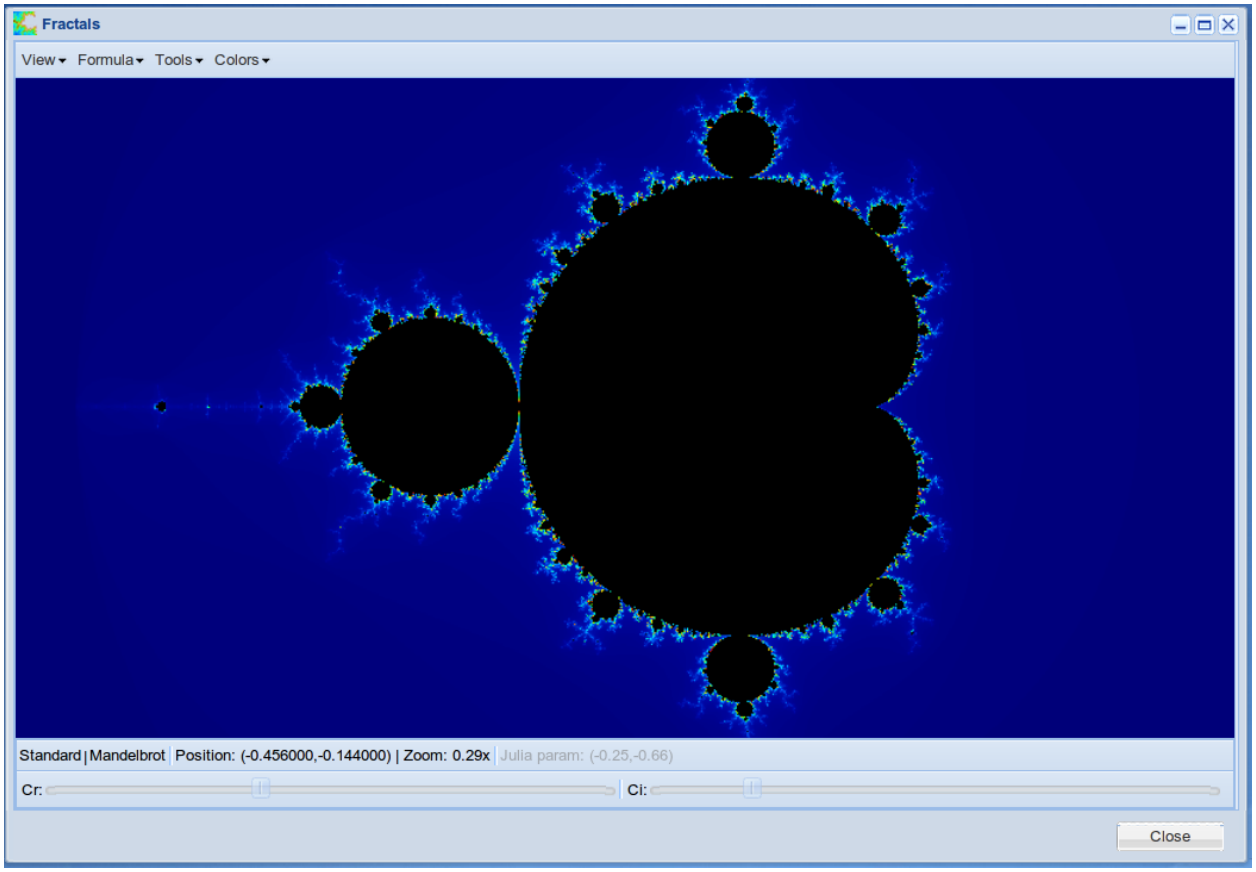

Figure 2: Fractal Explorer window with the Mandelbrot set.

Zooming in and out is done via the mouse wheel as shown in Fig. 3.

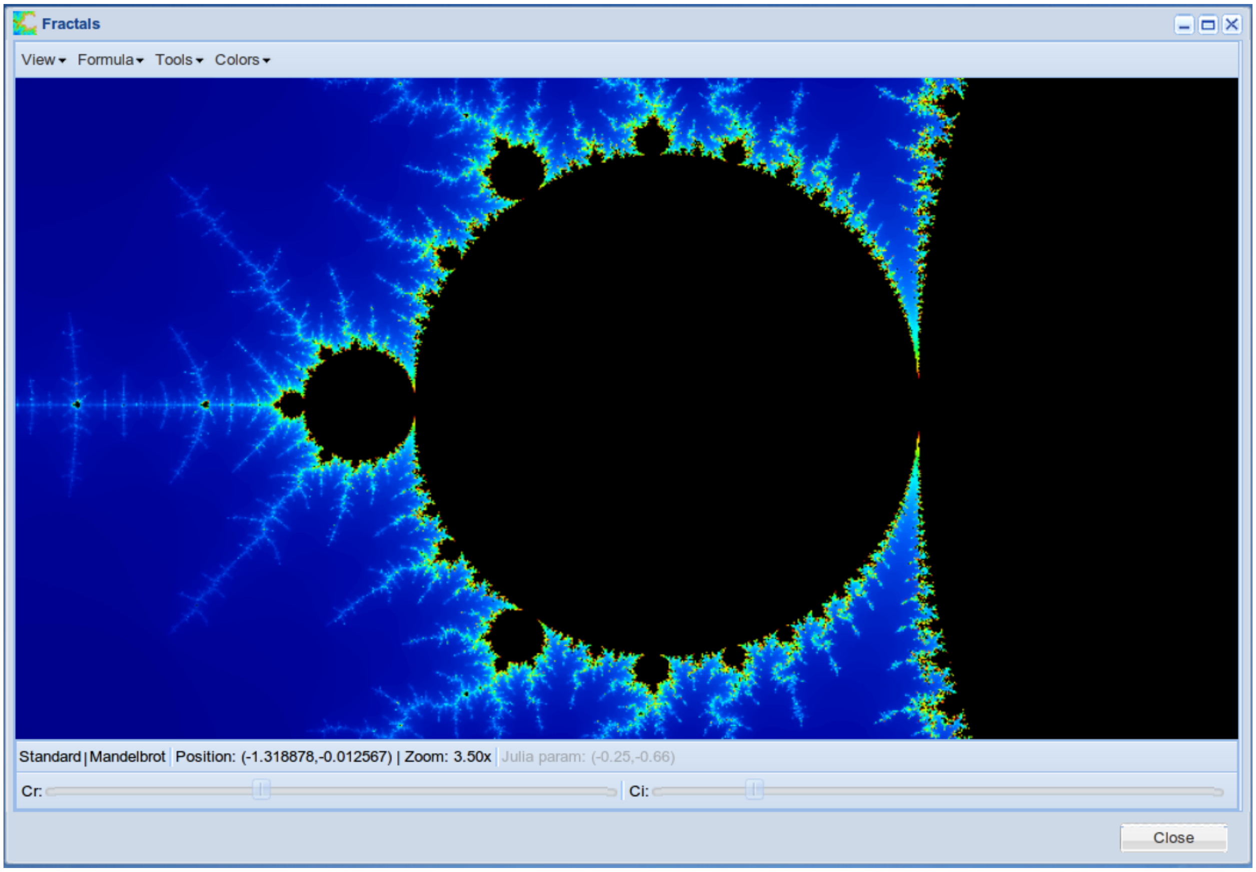

Figure 3: Zooming in reveals the self-similar fractal structure.

The menubar contains four items: *View*, *Formula*, *Tools* and *Colors*.

2.1 View Menu

The View menu allows the user to switch between the Standard and Artistic views. When Standard view is selected, the fractal is calculated using the standard iteration formula that was described earlier. With Artistic view, the Monte Carlo method is used to calculate the fractal, which has some interesting visual effects.

2.2 Formula Menu

Here the user can select between the Mandelbrot and Julia sets. If the latter is selected, two sliders in the bottom part of the window are activated, as shown in Fig. 4.

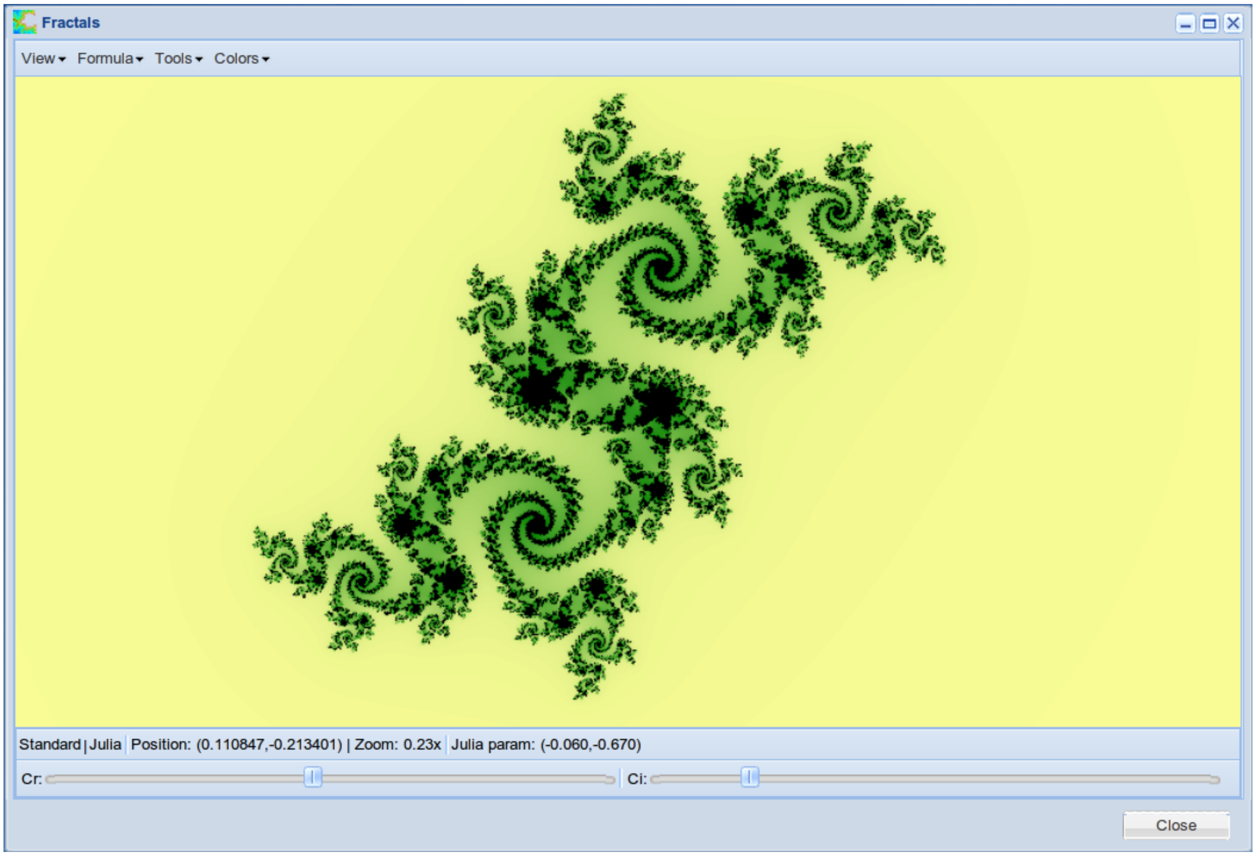

Figure 4: Sample Julia fractal. Constants c_r and c_i can be adjusted using sliders.

The sliders represent the real and imaginary parts c_r, c_i of the complex constant $c = c_r + ic_i$ in the formula $f(z) = z^2 + c$. When they are moved, the Julia fractal changes instantly. Of course not all values of the Julia constants are available through the sliders. This brings us to the *Tools* menu.

2.3 Tools Menu

This menu can be used to

- Change the center and zoom of the fractal view.
- Set the constants c_r and c_i manually.
- Reset the center and zoom to initial values.

2.4 Colors Menu

In the Colors menu one can choose between the default palette and a wide range of linear gradients. Fig. 5 shows the linear gradient menu.

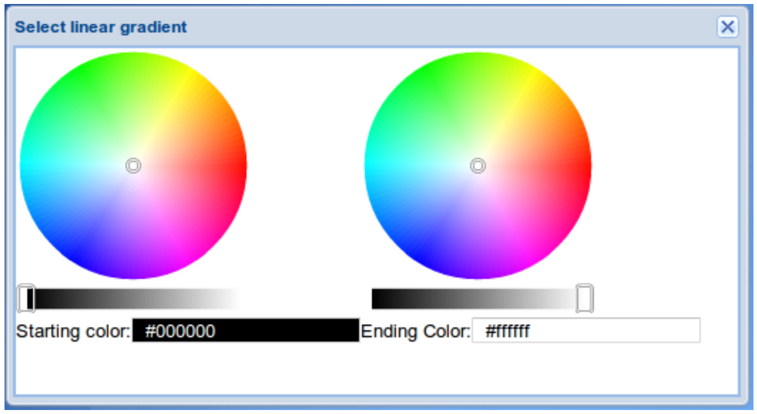

Figure 5: Color menu for linear gradients.

The start and end colors for the linear interpolation are selected by clicking into the left and right circles, respectively. Also use the bars to choose the level of darkness.